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DECOMPOSITION OF BINARY MATROIDS

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We prove results relating to the decomposition of a binary matroid, including its uniqueness when the matroid is cosimple. We extend the idea of "freedom" of an element in a matroid to "freedom" of a set, and show that there is a unique maximal integer polymatroid inducing a given binary matroid.

1. Introduction; balanced sets

The notion of a balanced set is a useful tool in analysing the subject of sums of matroids. Following [2], when a matroid $\mathscr{E}$ (with rank function ϱ) is induced by an *integer polymatroid* μ (i.e. an increasing integer-valued submodular set function such that $\mu(\emptyset)=0$), we say a set $A\subseteqq E$ is *μ-balanced* if $\mu A=\varrho A$. We say that a subset of E is *balanced* if it is so implied by the following rules:

B(1) circuits of $\mathscr{E}$ are balanced,
B(2) a union of balanced sets is balanced,
B(3) $[A]$ is balanced if and only if A is balanced,
B(4) if $A\cup B\in\mathscr{E}$ and A, B are balanced then so is $A\cap B$.

It is shown in [2] that A is balanced if and only if A is μ-balanced for every choice of μ, and that if we replace "balanced" by "μ-balanced" then B(1) to B(4) remain true. If $\mathscr{E}$ is a sum (or union) of matroids, $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$, say, where $\mathscr{F}$ and $\mathscr{G}$ have rank functions σ and τ, then $\mathscr{E}$ is induced by $\mu=\sigma+\tau$, and we can apply any result on balanced sets to this situation. Thus any balanced singleton in $\mathscr{E}$ is a loop of either $\mathscr{F}$ or $\mathscr{G}$. If every singleton of $\mathscr{E}$ is balanced then either $\mathscr{E}$ is disconnected and $\mathscr{F}\vee\mathscr{G}$ is a direct sum decomposition of $\mathscr{E}$, or $\mathscr{E}$ is *irreducible* (i.e. either $\mathscr{F}$ or $\mathscr{G}$ equals $\mathscr{E}$). This is dealt with in Duke [4], Ch 5 (where $\|e\|\leqq 1$ if and only if $\{e\}$ is balanced).

Throughout the paper, $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ will denote matroids (precisely, the independent set collections of matroids) on a finite set E, with rank functions ϱ, σ and τ respectively. We define the *support* of $\mathscr{F}$ to be $s(\mathscr{F})=\{f\in E\colon \{f\}\in\mathscr{F}\}$. A 1-coflat

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is a coflat of corank 1, and a 2-cocircuit is a cocircuit with 2 elements. Restrictions (deletions) are denoted by $\mathcal{E}|E\setminus A$ or $\mathcal{E}\setminus A$, and contractions by $\mathcal{E}\cdot E\setminus A$ or $\mathcal{E}/A$, as convenient. To say $\mathcal{E}$ is connected will mean that $\mathcal{E}|s(\mathcal{E})$ is connected (i.e. not separable). That is, a connected matroid may have loops. We start with several lemmas which should be of interest in their own right. The first is Theorem 3 of Lovász & Recski [5].

Lemma 1.1. *Let $\mathcal{E}=\mathcal{F}\vee\mathcal{G}$ (with rank functions ϱ, σ and τ). Let $\mathcal{E}=\mathcal{E}|A\oplus\mathcal{E}|B$. Then if $\mathcal{E}$ is coloop-free (or, generally, if A and B are $(\sigma+\tau)$-balanced) then $\mathcal{F}=\mathcal{F}|A\oplus\oplus\mathcal{F}|B$ and $\mathcal{G}=\mathcal{G}|A\oplus\mathcal{G}|B$.*

Proof. If $\mathcal{E}$ is coloop-free, then each connected component, and hence A and B, are fully dependent and so $(\sigma+\tau)$-balanced. Thus $\varrho E=\varrho A+\varrho B=\sigma A+\tau A+\sigma B+ +\tau B\geqq\sigma E+\tau E\geqq\varrho E$. So we have equality throughout, and $\sigma A+\sigma B=\sigma E$ and $\tau A+\tau B=\tau E$, whence the result. ∎

Lemma 1.2. (i) *Let the integer polymatroid μ induce $\mathcal{E}$ on E, and let $A\subseteqq E$. Let μ_A, given by $\mu_A(F)=\mu(F\cup A)-\mu A$, induce $\mathcal{F}$ on E. Then $\mathcal{F}$ is a strong map image of $\mathcal{E}/A$, and if A is μ-balanced, then $\mathcal{F}=\mathcal{E}/A$.*

(ii) *Let $\mathcal{E}=\mathcal{F}\vee\mathcal{G}$ and $A\subseteqq E$. Then $\mathcal{F}/A\vee\mathcal{G}/A$ is a strong map image of $\mathcal{E}/A$, and if A is $(\sigma+\tau)$-balanced, then $\mathcal{E}/A=\mathcal{F}/A\vee\mathcal{G}/A$.*

Proof. (i) Let $\mathcal{E}/A$ and $\mathcal{F}$ have rank functions $\varrho_A(\varrho_A(F)=\varrho(F\cup A)-\varrho A)$ and σ. Let $F\subseteqq G\subseteqq E\setminus A$; then for some $F'\subseteqq F$, $G'\subseteqq G$ and $A'\subseteqq A$,

$$\varrho_A(G)+\sigma F=\mu(G'\cup A')+|G\setminus G'|+|A\setminus A'|-\varrho A+\mu(F'\cup A)+|F\setminus F'|-\mu A$$

$$\geqq\mu(G'\cup F'\cup A)+\mu((G'\cap F')\cup A')+|G\setminus(G'\cup F')|+|F\setminus(G'\cap F')|$$

$$+|A\setminus A'|-\mu A-\varrho A\geqq\sigma G+\varrho_A(F).$$

Thus $\mathcal{F}$ is a strong map image of $\mathcal{E}/A$. If A is μ-balanced, then for $F\subseteqq E\setminus A$,

$$\varrho_A(F)=\varrho(F\cup A)-\varrho A\leqq\mu(F\cup A)-\mu A=\mu_A(F).$$

So $\mathcal{E}/A\subseteqq\mathcal{F}$ and we have the required equality.

(ii) Let $\mu=\sigma+\tau$; then $\mu_A=\sigma_A+\tau_A$ induces $\mathcal{F}/A\vee\mathcal{G}/A$. The result now follows from (i). ∎

We now record known results which will be used frequently later.

Lemma 1.3. *Let $\mathcal{E}\setminus e$ be disconnected, $\mathcal{E}\setminus e=\mathcal{E}|E_1\oplus\mathcal{E}|E_2$. Let $\mathcal{E}_i=\mathcal{E}/E_i$ $(i=1,2)$. Then*

(i) $\mathcal{E}=\mathcal{E}_1\vee\mathcal{E}_2$

(ii) *$\mathcal{E}$ is connected $\Rightarrow\mathcal{E}_1$ and $\mathcal{E}_2$ are connected, and if $\{e\}\in\mathcal{E}_1\cap\mathcal{E}_2$ then the reverse implication holds.*

(iii) *$\mathcal{E}$ is binary $\Leftrightarrow\mathcal{E}_1$ and $\mathcal{E}_2$ are binary.*

Proof. (i) If $\mathcal{E}$ is connected, this is [1], Theorem 4; otherwise, the result follows by considering the connected component of $\mathcal{E}$ containing e.

(ii) Let $\mathscr{E}_1 = \mathscr{E}_1|A \oplus \mathscr{E}_1|B$, with $e \in B$. Then $\mathscr{E} = \mathscr{E}_1|A \oplus (\mathscr{E}_1|B \vee \mathscr{E}_2)$, and the result follows. Conversely suppose $\mathscr{E}$ is disconnected; let $\mathscr{E} = \mathscr{E}|A \oplus \mathscr{E}|B$, where $e \in A$, B is connected and both summands are non-trivial. As B is connected, we may assume that $B \subseteq E_1$. Then $\mathscr{E}/E_2 = (\mathscr{E}|A)/E_2 \oplus \mathscr{E}|B$; if $\{e\} \in (\mathscr{E}|A)/E_2$, $\mathscr{E}/E_2$ is disconnected.

(iii) $\Rightarrow$ is well known. A proof of $\Leftarrow$ appears within the proof of [1], Theorem 6. Alternatively, it is easy to construct a representation of $\mathscr{E}$ over GF(2), given representations of $\mathscr{E}_1$ and $\mathscr{E}_2$. (By such a proof, a corresponding result is true for representability over any field.) ∎

2. Decomposition of binary matroids

We first use the theory of balanced sets to give a shorter proof of a result which appears in [4], Lemma 4.4, and is close to [1], Theorem 5.

Lemma 2.1. *Let $\mathscr{E}$ be a coloop-free binary matroid on E, and let $\{e\} \in \mathscr{E}$ such that $\mathscr{E} \setminus e$ is connected. Then $\{e\}$ is balanced.*

Proof. Suppose that $\{e\}$ is not balanced. Let A be a maximal independent set which contains e but does not contain any balanced set containing e. Since $\mathscr{E}$ is coloop-free, E and every basis of $\mathscr{E}$ are balanced by B(1) and B(2); thus $\varrho(A) \leqq \mathrm{rk}(\mathscr{E}) - 1$. Suppose $\varrho(A) \leqq \mathrm{rk}(\mathscr{E}) - 2$. Then let $A \cup \{a, b\} \in \mathscr{E}$; by the maximality of A, there are balanced sets $A' \cup a$ and $A'' \cup b$, where $e \in A'$, $A'' \subseteq A$. Then as $(A' \cup a) \cup (A'' \cup b) \in \mathscr{E}$, $(A' \cup a) \cap (A'' \cup b) = A' \cap A''$ is balanced, by Lemma 1.4(iv), contrary to the choice of A. Thus $\varrho(A) = \mathrm{rk}(\mathscr{E}) - 1$.

Now let $F = [A]$ (the span of A), $D = E \setminus F$ (which is a cocircuit) and $F' = F \setminus e$. Suppose that $P \subseteq F$ is balanced. We show $e \notin P$. Let $Q \subseteq A$, where Q is minimal such that $P \subseteq [Q]$. Then $P \cup Q$ is a union of P and some circuits, and hence is balanced; as $[P \cup Q] = [Q]$, Q is balanced. As $P \cap A \subseteq Q \subseteq A$, $e \notin P$. Thus F does not contain a balanced set containing e; hence e is a coloop of F.

As $\mathscr{E}$ is binary it has at most 3 hyperplanes containing the coline F'; since $E \setminus e$ spans $\mathscr{E}$, it has exactly 3, say $F' \cup e$, $F' \cup G$ and $F' \cup H$. Thus $G \cup e$, $H \cup e$ and $G \cup H = D$ are cocircuits of $\mathscr{E}$.

As $\mathscr{E} \setminus e$ is connected, we may choose $g \in G$, $h \in H$ and a circuit C containing g and h but not e; choose g, h and C so that $|C \cap D|$ is minimal. As a circuit and a cocircuit do not intersect in exactly one element (this fact will be used frequently), let $g' \in C \cap (G \cup e) = C \cap G$, such that $g' \neq g$.

Let B be a basis of F containing $C \cap F$; for $x \notin B \cup g$, let $C(x)$ be the fundamental circuit of x with respect to the basis $B \cup g$ of $\mathscr{E}$. As $|C(g') \cap D| \neq 1$, $C(g') \cap D = \{g, g'\}$, and as $|C(g') \cap (H \cup e)| \neq 1$, $e \notin C(g')$. As $\mathscr{E}$ is binary, the symmetric difference $C(g') \triangle C$ contains a circuit C_H containing h. By the choice of C, $C_H \cap G = \emptyset$. Clearly $C_H \cap C(g') \neq \emptyset$, so $C_H \cup C(g')$ contains a circuit C' containing g and h; we have $|C' \cap G| = |\{g, g'\}| = 2$ and $C' \cap H \subseteq C \cap H$. Thus, by the choice of C, $|C \cap G| = 2$; similarly it can be shown that $|C \cap H| = 2$. We have $C \cap G = \{g, g'\}$; let $C \cap H = \{h, h'\}$. As $C_H \subseteq C \triangle C(g')$ and $C_H \cap G = \emptyset$, $C_H \cap D = \{h, h'\}$; hence $C_H = C(h) \triangle C(h')$ and $C = C(g') \triangle C_H$. Let $f \in C(g') \cap C_H$, and without loss of generality let $f \in C(h') \setminus C(h)$.

We have balanced sets $C(h) \setminus h$ and $(C(g') \triangle C(h')) \setminus g'$ (by Lemma 1.4), neither of which contains f. Thus $(C(h) \setminus h) \cup \{(C(g') \triangle C(h')) \setminus g'\}$ does not contain g', h or $C(h')$, and hence is independent. Therefore, by Lemma 1.4(iv), $(C(h)\setminus h)\cap\{(C(g')\triangle C(h'))\setminus g'\}$ is a balanced subset of F, containing e (since each circuit $C(x)$ satisfies $|C(x)\cap(H\cup e)|\neq 1$). This contradicts our choice of F; hence $\{e\}$ is balanced. ∎

The next theorem confirms a conjecture of Recski ([9], Problem 4).

Theorem 2.2. *Let $\mathscr{E}$ be a cosimple binary matroid, and suppose that $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$. Then $\mathscr{F}$ and $\mathscr{G}$ are also binary cosimple.*

Proof. It is easy to see that $\mathscr{F}$ and $\mathscr{G}$ must be cosimple. We prove that they are binary by induction on $|E|$ (i.e. $|s(\mathscr{E})|$). If $s(\mathscr{F})\cap s(\mathscr{G})=\emptyset$, then the result is trivial, otherwise let $e\in s(\mathscr{F})\cap s(\mathscr{G})$. As $\{e\}$ is not balanced, we have $\mathscr{E}\setminus e=\mathscr{E}|E_1\oplus\mathscr{E}|E_2$ and $\mathscr{E}=\mathscr{E}/E_2\vee\mathscr{E}/E_1$, by Lemmas 2.1 and 1.3. As $\mathscr{E}\setminus e$ is coloop-free, $\mathscr{F}\setminus e=\mathscr{F}|E_1\oplus\mathscr{F}|E_2$ (by Lemma 1.1), and so $\mathscr{F}=\mathscr{F}/E_2\vee\mathscr{F}/E_1$ (similarly for $\mathscr{G}$). Also, E_1 and E_2 are fully dependent and hence balanced (in $\mathscr{E}$), and so $\mathscr{E}/E_i=\mathscr{F}/E_i\vee\mathscr{G}/E_i$ $(i=1, 2)$, by Lemma 1.2(ii). Now $\mathscr{E}/E_i$ are binary cosimple, being contractions of $\mathscr{E}$, and so, by induction on $|E|$, $\mathscr{F}/E_i$ and $\mathscr{G}/E_i$ are also binary. Hence, by Lemma 1.3 (iii), $\mathscr{F}$ and $\mathscr{G}$ are binary. ∎

We now proceed towards proving the uniqueness of the sum decomposition of a binary matroid.

Theorem 2.3. *Let $\mathscr{E}$ be a binary coloop-free matroid, and suppose that $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$, where $\mathscr{F}$ and $\mathscr{G}$ are connected. Let $A=s(\mathscr{F})\cap s(\mathscr{G})$. If $|A|\geqq 2$, then A is a union of 2-cocircuits of $\mathscr{E}$.*

Proof. Suppose the result is false; look at a counterexample with $|E|$ (i.e. $|s(\mathscr{E})|$) minimal. Let $e\in A$ such that e is not in a 2-cocircuit in A, that is, $\mathscr{E}\setminus e$ has no coloops in A.

As $(\sigma+\tau)(e)=2$, $\{e\}$ is not balanced, and so, by Lemma 2.1, $\mathscr{E}\setminus e$ is disconnected and we have $\mathscr{E}\setminus e=\mathscr{E}|E_1\oplus\mathscr{E}|E_2$, where E_1 and E_2 are non-empty unions of connected components of $\mathscr{E}\setminus e$. As coloops of $\mathscr{E}\setminus e$ are in $E\setminus A$ and hence are $(\sigma+\tau)$-balanced, each connected component of $\mathscr{E}\setminus e$ is $(\sigma+\tau)$-balanced and so E_1 and E_2 are $(\sigma+\tau)$-balanced.

Now as E_1 and E_2 are $(\sigma+\tau)$-balanced, $\mathscr{E}/E_i=\mathscr{F}/E_i\vee\mathscr{G}/E_i$ $(i=1, 2)$, by Lemma 1.2(ii). Also $\mathscr{F}\setminus e=\mathscr{F}|E_1\oplus\mathscr{F}|E_2$, and so $\mathscr{F}=\mathscr{F}/E_2\vee\mathscr{F}/E_1$, by Lemmas 1.1 and 1.3. As $\mathscr{F}$ is connected, $\mathscr{F}/E_i$ are connected $(i=1, 2)$. Corresponding results apply for $\mathscr{G}$. As $\mathscr{E}/E_i=\mathscr{F}/E_i\vee\mathscr{G}/E_i$, we have, by the minimality of $|E|$, that $s(\mathscr{F}/E_i)\cap s(\mathscr{G}/E_i)$ is either a singleton or is a union of 2-cocircuits (of $\mathscr{E}/E_i$, and hence also of $\mathscr{E}$).

Now $s(\mathscr{F}/E_1)$ is equal to either $s(\mathscr{F})\cap(E_2\cup e)$ or $s(\mathscr{F})\cap E_2$. Suppose it is the latter. Then E_1 spans e, $\mathscr{F}=\mathscr{F}|(E_1\cup e)\oplus\mathscr{F}|E_2$ and, since $\mathscr{F}$ is connected, $\mathscr{F}|E_2$ is null and $\mathscr{F}|(E_1\cup e)=\mathscr{F}/E_2=\mathscr{F}$. Thus $\mathscr{E}=\mathscr{F}\vee\mathscr{G}/E_2\vee\mathscr{G}/E_1$. If also $e\notin s(\mathscr{G}/E_2)$, then by a similar argument, $\mathscr{G}|E_1$ is null and $A=\{e\}$; otherwise $A=$ $=s(\mathscr{F})\cap s(\mathscr{G}/E_2)$ which, as above, is either a singleton or a union of 2-cocircuits of $\mathscr{E}$.

We may now assume that $s(\mathscr{F}/E_i)$ and $s(\mathscr{G}/E_i)$ all contain e. Then $s(\mathscr{F}/E_i)\cap$ $\cap s(\mathscr{G}/E_i)$ $(i=1, 2)$ are each either equal to $\{e\}$ or a union of 2-cocircuits of $\mathscr{E}$. Hence the same is true of their union, which is equal to A. ∎

We conjecture that, under the assumptions of Theorem 2.3, $\varrho^*(A)=1$ (i.e. every 2-subset of A is a cocircuit of $\mathscr{E}$).

We now can deduce the uniqueness of the sum decomposition of cosimple binary matroids, as conjectured by Cunningham [1], §6.

Definition. $\mathscr{E}=\bigvee_{i\in I}\mathscr{E}_i$ is a *forest decomposition* of $\mathscr{E}$ if the bipartite graph $\big(E\cup I, \{\{e,i\}: \{e\}\in\mathscr{E}_i\}\big)$ is a forest.

Corollary 2.4. (i) ([1], Conjecture 1). *Every sum decomposition of a binary cosimple matroid into connected matroids is a forest decomposition.*

(ii) ([1], Conjecture 2). *Every cosimple binary matroid has a unique sum decomposition into irreducible matroids.*

Proof. (i) Suppose otherwise, say $\mathscr{E}=\mathscr{E}_1\vee\ldots\vee\mathscr{E}_n$, where each $\mathscr{E}_i$ is connected and there are distinct elements $e_1, e_2, \ldots, e_m$ $(m\leqq n)$ such that $\{e_i\}\in\mathscr{E}_i\cap\mathscr{E}_{i+1}$ $(i=1,\ldots,m-1)$ and $\{e_m\}\in\mathscr{E}_m\cap\mathscr{E}_1$. By theorem 2.2, $\mathscr{E}_1\vee\ldots\vee\mathscr{E}_m$ and $\mathscr{E}_2\vee\ldots\vee\mathscr{E}_m$ are binary cosimple; as the latter is coloop-free, it follows from Lemma 1.1 that it is connected. Thus $\mathscr{E}_1\vee(\mathscr{E}_2\vee\ldots\vee\mathscr{E}_m)$ is binary cosimple (and so has no 2-cocircuits) and $\{e_1,e_m\}\subseteq s(\mathscr{E}_1)\cap s(\mathscr{E}_2\vee\ldots\vee\mathscr{E}_m)$; this contradicts Theorem 2.3.

(ii) This follows from (i) as shown by Cunningham ([1], §6). ▮

Partial confirmation of a conjecture of Recski ([9], Problem 6) also follows; we first need an easy lemma.

Lemma 2.5. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be graphic, with $|s(\mathscr{E}_1)\cap s(\mathscr{E}_2)|\leqq 1$. Then $\mathscr{E}_1\vee\mathscr{E}_2$ is graphic.*

Proof. The result is trivial if $s(\mathscr{E}_1)\cap s(\mathscr{E}_2)=\emptyset$. Let $s(\mathscr{E}_i)=E_i\cup e$ $(i=1,2)$, where $E_1\cap E_2=\emptyset$. Let $\mathscr{E}_i$ be represented by the graph $(V_i, E_i\cup e_i)$ (where e_i represents e in $\mathscr{E}_i$, and $V_1\cap V_2=\emptyset$), and let v_i, w_i be the endpoints of e_i $(i=1,2)$. Now take $(V_1\cup V_2, E_1\cup E_2)$, identify the vertices v_1 and v_2 and add an edge e joining w_1 to w_2. It can be checked that the resulting graph represents $\mathscr{E}_1\vee\mathscr{E}_2$. ▮

Corollary 2.6. *Let $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$ be binary and cosimple, and let $\mathscr{F}$ and $\mathscr{G}$ be graphic. Then $\mathscr{E}$ is graphic.*

Proof. Decompose $\mathscr{F}$ and $\mathscr{G}$ into connected components (each of which is graphic), getting $\mathscr{F}=\bigoplus_{i\in I}\mathscr{F}_i$ and $\mathscr{G}=\bigoplus_{j\in J}\mathscr{G}_j$. By Corollary 2.4(i), $\mathscr{E}=\bigvee\big(\{\mathscr{F}_i: i\in I\}\cup\{\mathscr{G}_j: j\in J\}\big)$ is a forest decomposition. Thus, by repeated application of Lemma 2.5, $\mathscr{E}$ is graphic. ▮

3. Freedom in binary matroids

In [4], Duke defines the "freedom" $\|a\|$, or $\|a\|_{\mathscr{E}}$, of an element a in a matroid $\mathscr{E}$, and shows that $\|a\|$ is equal to the maximum value of $\mu(a)$ for an integer polymatroid μ (i.e. an integer valued submodular increasing set function with $\mu(\emptyset)=0$) which induces $\mathscr{E}$ (see [7]). In [3] we generalize this to a definition of $\|A\|$ for $A\subseteq E$, and likewise show that $\|A\|$ is the maximum value of $\mu(A)$ for an integer polymatroid μ inducing $\mathscr{E}$. The following extension of [4], Theorem 5.3, appears in [3].

Lemma 3.1. *Let* $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$, *and* $F\subseteq E$. *Then* $\|F\|_{\mathscr{E}}\geqq\|F\|_{\mathscr{F}}+\|F\|_{\mathscr{G}}$.

Proof. Let ν, ξ induce $\mathscr{F}$, $\mathscr{G}$ respectively. Then, by [7], Theorem 6.5, $\nu+\xi$ induces $\mathscr{F}\vee\mathscr{G}$. ▮

Several examples (eg in [6]) have been given to show that there is not in general a unique maximal integer polymatroid inducing a given matroid. We now show that for binary matroids, however, there is. We need some lemmas.

Lemma 3.2. *Let* $\mathscr{E}$ *be a cosimple matroid, such that* $\mathscr{E}\setminus e$ *is disconnected, say* $\mathscr{E}\setminus e=\mathscr{E}|E_1\oplus\mathscr{E}|E_2$. *Let* $\mathscr{E}_i=\mathscr{E}.E_i\cup e$ (*so* $\mathscr{E}=\mathscr{E}_1\vee\mathscr{E}_2$). *Then, for* $F\subseteq E$, $\|F\|_{\mathscr{E}}=\|F\|_{\mathscr{E}_1}+\|F\|_{\mathscr{E}_2}$.

Proof. Let μ induce $\mathscr{E}$ such that $\mu(\mathscr{F})=\|F\|_{\mathscr{E}}$. Since $\mathscr{E}\setminus e$ is coloop-free, E_1 and E_2 are fully dependent and hence balanced, and so $\mathscr{E}_1$ is induced by μ_1, given by $\mu_1(A)=\mu(A\cup E_2)-\mu(E_2)$ (and likewise $\mathscr{E}_2$ by μ_2), by Lemma 1.2(i). Thus

$$\begin{aligned}\|F\|_{\mathscr{E}_1}+\|F\|_{\mathscr{E}_2}&\geqq\mu_1(F)+\mu_2(F)=\mu(F\cup E_1)+\mu(F\cup E_2)-\mu(E_1)-\mu(E_2)\\&\geqq\mu(F\cup E_1\cup E_2)+\mu(F)-\mu(E_1)-\mu(E_2)\\&=\varrho(E)+\mu(F)-\varrho(E_1)-\varrho(E_2)\end{aligned}$$

(since E, E_1 and E_2 are fully dependent and so balanced)

$$=\mu(F)=\|F\|_E.$$

By Lemma 3.1 we have equality. ▮

Lemma 3.3. *Let* $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$, *where* $\mathscr{E}$ *is coloop-free,* A *is a* 1-*coflat,* $a\in A$, $A'=A\setminus a$, $\mathscr{F}=\mathscr{E}/A'$ *and* $\mathscr{G}=\mathscr{E}.A$. *Then for* $F\subseteq E$ *such that either* $a\in F$ *or* $A\cap F=\emptyset$, $\|F\|_{\mathscr{E}}=\|F\|_{\mathscr{F}}+\|F\|_{\mathscr{G}}$.

Proof. Let μ induce $\mathscr{E}$ such that $\mu(F)=\|F\|_{\mathscr{E}}$. Define ν on $E\setminus A'$ by

$$\nu(G)=\mu(G)\quad(a\notin G)$$
$$\nu(G)=\mu(G\cup A)-|A'|\quad(a\in G).$$

We need to show that ν is increasing. Let $H\subseteq E\setminus A$; we must show that $\nu(H)\leqq\nu(H\cup a)$. As A is a coflat, $E\setminus A$ is fully dependent and so balanced; hence

$$\begin{aligned}\nu(H)=\mu(H)&\leqq\mu(E\setminus A)+\mu(H\cup A)-\mu(E)\quad\text{(by submodularity)}\\&=\varrho(E\setminus A)-\varrho(E)+\mu(H\cup A)\\&=\mu(H\cup A)-|A'|\quad(\text{since }\varrho^*(A)=1\text{ and }\mathscr{E}\text{ is coloop-free})\\&=\nu(H\cup a).\end{aligned}$$

It is easy to see that ν is submodular.

We show that ν induces $\mathscr{F}(=\mathscr{E}/A')$. Let C be a circuit of $\mathscr{F}$. If $a\notin C$, then C is a circuit of $\mathscr{E}$ and $\nu(C)=\mu(C)=\varrho(C)<|C|$. If $a\in C$, then $C\cup A$ is a circuit of $\mathscr{E}$ (since elements of A are in series in $\mathscr{E}$), and then $\nu(C)=\mu(C\cup A)-|A'|=$

$=\varrho(C\cup A)-|A'|=|C|-1$. Now let $H\subseteq E\setminus A'$; if $a\in H$, then

$$\sigma(H)=\varrho(H\cup A')-\varrho(A')\leqq\mu(H\cup A)-|A'|=\nu(H),$$

and if $a\notin H$, then $\sigma(H)\leqq\varrho(H)\leqq\mu(H)=\nu(H)$. Thus ν induces $\mathscr{F}$.

Let $\mathscr{G}$, which is a single circuit A, be induced by $\xi(B)=|A'|$ for $\emptyset\subset B\subseteq A$. Then if $a\in F$, $(\nu+\xi)(F)=\mu(F\cup A)$, and so if $a\in F$ or $F\cap A=\emptyset$, we have

$$\|F\|_{\mathscr{F}}+\|F\|_{\mathscr{G}}\geqq(\nu+\xi)(F)\geqq\mu(F)=\|F\|_{\mathscr{E}},$$

and the equality follows from Lemma 3.1 ∎

Lemma 3.4. *Let $\mathscr{E}=\bigvee_{i\in I}\mathscr{E}_i$, and let A be a 1-coflat of $\mathscr{E}$. For $i\in I$, let μ_i induce $\mathscr{E}$ and define ν_i by*

$$\nu_i(F)=\mu_i(F)\qquad(F\cap A=\emptyset)$$
$$\nu_i(F)=\mu_i(F\cup A)(F\cap A\neq\emptyset).$$

Let $\mathscr{F}_i$ be induced by ν_i. Then $\mathscr{E}=\bigvee_{i\in I}\mathscr{F}_i$.

Proof. Clearly each ν_i is an integer polymatroid, and $\nu_i\geqq\mu_i$. Let C be a circuit of $\mathscr{E}$; then either $A\subseteq C$ or $C\cap A=\emptyset$ and so $\sum_{i\in I}\nu_i(C)=\sum_{i\in I}\mu_i(C)=\varrho(C)$. Thus $\sum\nu_i$ induces $\mathscr{E}$ (as does $\sum\mu_i$) and the result follows. ∎

Theorem 3.5. *Let $\mathscr{E}$ be a binary matroid. Then there is a unique maximal integer polymatroid μ inducing $\mathscr{E}$, given by $\mu(H)=\|H\|_{\mathscr{E}}$.*

Proof. Let $\{A_j\colon j\in J\}$ be the set of 1-coflats of $\mathscr{E}$ and for each $j\in J$, choose $a_j\in A_j$ and let $A_j'=A_j\setminus a_j$. Let $\mathscr{G}_j=\mathscr{E}.A_j$ and let $\mathscr{F}=\mathscr{E}/\bigcup_{j\in J}A_j'$. Then $\mathscr{E}=\mathscr{F}\vee\bigvee_{j\in J}\mathscr{G}_j$, where $\mathscr{F}$ is binary cosimple. Let $\mathscr{F}=\bigvee_{i\in I}\mathscr{F}_i$ be the unique decomposition of $\mathscr{F}$ into irreducible matroids $\mathscr{F}_i$ (by Corollary 2.4). Since it is a forest decomposition, and since each summand of $\mathscr{F}$ is binary cosimple (by Theorem 2.2), it follows from Lemma 3.2 that $\|F\|_{\mathscr{F}}=\sum_{i\in I}\|F\|_{\mathscr{F}_i}$ for $F\subseteq E$. Now since each $\mathscr{F}_i$ is irreducible it follows from Lemmas 1.3 and 2.1 that each singleton, and hence each set, is balanced in $\mathscr{F}_i$. Thus, $\sigma_i(H)$ (the rank of H in $\mathscr{F}_i$) is equal to $\|H\|_{\mathscr{F}_i}$; also, $\|\cdot\|_{\mathscr{G}_j}$ (given by $\|H\|=|A_j'|$ provided that $H\cap A_j\neq\emptyset$) induces G_j.

Hence $\sum_{i\in I}\|\cdot\|_{\mathscr{F}_i}+\sum_{j\in J}\|\cdot\|_{\mathscr{G}_j}$ is an integer polymatroid which induces $\mathscr{E}$; by Lemma 3.3 and the remarks above, it is equal to $\|\cdot\|_{\mathscr{E}}$, and therefore maximal, on sets H satisfying $H\cap A_j\neq\emptyset\Rightarrow a_j\in H(j\in J)$. Following Lemma 3.4 we now define $\mathscr{F}_i'$ from $\mathscr{F}_i$. Let $\psi(H)=H\cup\{a_j\colon H\cap A_j\neq\emptyset\}$, let $\sigma_i'(H)=\sigma_i(\psi(H))$, and let $\mathscr{F}_i'$ be induced by σ_i' (which is, in fact, its rank function). By Lemma 3.4, $\mathscr{E}=\bigvee_{i\in I}\mathscr{F}_i'\vee\bigvee_{j\in J}\mathscr{G}_j$, and so $\mu=\sum_{i\in I}\sigma_i'+\sum_{j\in J}\|\cdot\|_{\mathscr{G}_j}$ induces $\mathscr{E}$. Now, for $H\subseteq E$,

$$\mu(H)=\sum_{i\in I}\sigma_i(\psi(H))+\sum_{j\in J}\|H\|_{\mathscr{G}_j}$$
$$=\sum_{i\in I}\|\psi(H)\|_{\mathscr{F}_i}+\sum_{j\in J}\|\psi(H)\|_{\mathscr{G}_j}=\|\psi(H)\|_{\mathscr{E}}\geqq\|H\|_{\mathscr{E}}.$$

Therefore we have equality, and μ is maximal. ∎

We next consider the matroid equation $\mathscr{E}=\mathscr{F}\vee\mathscr{X}$, where $\mathscr{E}$ and $\mathscr{F}$ are given. Recski (private communication) conjectures that, when $\mathscr{E}$ is binary and the equation can be solved for $\mathscr{X}$, there is a unique maximal solution. He has proved this, in [8], for $\mathscr{E}$ graphic. First we show that it is not true for general $\mathscr{E}$.

Example 3.6. Let $E=\{1, 2, \ldots, 11\}$, and let $\mathscr{E}$ be of rank 5, with non-spanning circuits $\{1, 3, 4, 11\}$, $\{1, 2, 5, 6, 7\}$ and $\{1, 2, 8, 9, 10\}$. Let $\mathscr{F}$, $\mathscr{G}_1$ and $\mathscr{G}_2$ (with rank functions σ, τ_1 and τ_2) be the free-est matroids such that:

$$\operatorname{rk}(\mathscr{F})=2,\quad \sigma(\{1,3,4,11\})=1.$$

$$\operatorname{rk}(\mathscr{G}_1)=3,\quad \tau_1(\{1,3,4,11\})=\tau_1(\{1,5,6,7\})=\tau_1(\{1,8,9,10\})=2,\quad \tau_1(2)=0$$

$$\operatorname{rk}(\mathscr{G}_2)=3,\quad \tau_2(\{3,4,11\})=\tau_2(\{2,5,6,7\})=\tau_2(\{2,8,9,10\})=2,\quad \tau_2(1)=0.$$

Then it can be checked that $\mathscr{E}=\mathscr{F}\vee\mathscr{G}_1=\mathscr{F}\vee\mathscr{G}_2$. However, suppose $\mathscr{G}\supseteq\mathscr{G}_1\cup\mathscr{G}_2$, and $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$. Then $\tau(\{1,2\})+\tau(\{1,3,4\})\geqq\tau(1)+\tau(\{1,2,3,4\})\geqq 4$. As $\{1,3,4,11\}$ is a circuit of $\mathscr{E}$, and thus balanced, $\tau(\{1,3,4,11\})=2$, and so $\tau(\{1,2\})=2$. As $\{1,2,5,6,7\}$ is a circuit, $\tau(\{1,2,5,6\})\leqq 2$, so 5, 6 and likewise 8 are spanned by $\{1,2\}$ in $\mathscr{G}$. This gives $(\sigma+\tau)(\{1,2,5,6,8\})=2+2=4$, a contradiction. ▮

When $\mathscr{E}$ is cosimple, a result is immediate.

Corollary 3.7. *Let $\mathscr{E}$ be a binary cosimple matroid, and let $\mathscr{E}=\mathscr{F}\vee\mathscr{G}$. Then for $F\subseteqq E$, $\|F\|_{\mathscr{E}}=\|F\|_{\mathscr{F}}+\|F\|_{\mathscr{G}}$; also $\mathscr{G}$ is uniquely determined by $\mathscr{E}$ and $\mathscr{F}$, as it is induced by the integer polymatroid ξ, given by $\xi(F)=\|F\|_{\mathscr{E}}-\|F\|_{\mathscr{F}}$, for $F\subseteqq E$.*

Proof. By Theorem 2.2, $\mathscr{F}$ and $\mathscr{G}$ are binary cosimple. Let $\mathscr{F}=\bigvee_{i\in I}\mathscr{F}_i$ and $\mathscr{G}=\bigvee_{j\in J}\mathscr{G}_j$ be the unique (by Corollary 2.4(ii)) decompositions of $\mathscr{F}$ and $\mathscr{G}$ into irreducible matroids. Then $\mathscr{E}=\bigvee(\{\mathscr{F}_i\colon i\in I\}\cup\{\mathscr{G}_j\colon j\in J\})$ is the unique decomposition of $\mathscr{E}$ into irreducible matroids. It follows from the proof of Theorem 3.5 that $\|F\|_{\mathscr{F}}=\sum_{i\in I}\|F\|_{\mathscr{F}_i}$, $\|F\|_{\mathscr{G}}=\sum_{j\in J}\|F\|_{\mathscr{G}_j}$ and $\|F\|_{\mathscr{E}}=\sum_{i\in I}\|F\|_{\mathscr{F}_i}+\sum_{j\in J}\|F\|_{\mathscr{G}_j}$. Also by Theorem 3.5, $\mathscr{G}$ is induced by $\|\cdot\|_{\mathscr{G}}=\|\cdot\|_{\mathscr{E}}-\|\cdot\|_{\mathscr{F}}$. ▮

We now look at the case where $\mathscr{E}$ is not necessarily cosimple, for which we can obtain a partial result in this direction.

Lemma 3.8. *Let $\mathscr{E}=\bigvee_{i\in I}\mathscr{E}_i$, where $\mathscr{E}$ is coloop-free; let A be a 1-coflat of $\mathscr{E}$, $a\in A$ and $A'=A\setminus a$. For $i\in I$, let $\mu_i=\varrho_i$ and define $\mathscr{F}_i$ as in Lemma 3.4. Then $\mathscr{E}\cdot A=\bigvee_{i\in I}\mathscr{F}_i\cdot A$, of which A' is a basis; let $A'=\bigcup_{i\in I}B_i$ (disjoint union) such that $B_i\in\mathscr{F}_i\cdot A$ for each $i\in I$. Let $\mathscr{F}_i'$ denote $\mathscr{F}_i/B_i|(E\setminus A')$. Then $\mathscr{E}/A'=\bigvee_{i\in I}\mathscr{F}_i'$.*

Proof. As A is a coflat, $E\setminus A$ is fully dependent and so balanced; as $\mathscr{E}=\bigvee_{i\in I}\mathscr{F}_i$ (by Lemma 3.4), it follows from Lemma 1.2(ii) that $\mathscr{E}\cdot A=\bigvee_{i\in I}\mathscr{F}_i\cdot A$. Clearly $\mathscr{E}/A'\supseteqq\bigvee_{i\in I}\mathscr{F}_i'$. Let $H\in\mathscr{E}/A'$, such that $a\notin H$, and let $H=\bigcup_{i\in I}H_i$, where $H_i\in\mathscr{F}_i$.

Then for each $i\in I$, $H_i\cup B_i\in\mathcal{F}_i$, and $H_i\in\mathcal{F}_i'$ and so $H\in\bigvee_{i\in I}\mathcal{F}_i'$. If $H\cup a\in\mathcal{E}/A'$ $(a\notin H)$, we have $H\cup A\in\mathcal{E}=\bigvee_{i\in I}\mathcal{F}_i$, say $H_i\cup C_i\in\mathcal{F}_i$, where $H=\bigcup_{i\in I}H_i$ and $A=\bigcup_{i\in I}C_i$. Choose $k\in I$ such that $|C_k|>|B_k|$; as $\mathcal{F}_k\supseteq\mathcal{E}_k$ and all elements of A are equivalent in $\mathcal{F}_k$, $H_k\cup a\cup B_k\in\mathcal{F}_k$ and $H_k\cup a\in\mathcal{F}_k'$. As before, $H_i\in\mathcal{F}_i'$ for all $i\in I$. Thus $H\cup a\in\bigvee_{i\in I}\mathcal{F}_i'$, and the result follows. ∎

Theorem 3.9. *Let $\mathcal{E}=\mathcal{F}\vee\mathcal{G}$, where $\mathcal{E}$ is binary and coloop-free. Let $\mathcal{G}'$ be derived from $\mathcal{G}$ as follows: using every* 1*-coflat of $\mathcal{E}$ in turn, repeatedly apply the construction which, in Lemma* 3.8, *derives $\mathcal{F}_i'$ from $\mathcal{E}_i$ using the coflat A. Then $\mathcal{G}'$ is binary cosimple, and is determined by $\mathcal{E}$ and $\mathcal{F}$.*

Proof. Let $\{A_j\colon j\in J\}$ be the set of 1-coflats of $\mathcal{E}$, and for each $j\in J$, choose $a_j\in A_j$ and let $A_j'=A_j\setminus a_j$. Then, by repeated application of Lemma 3.8, $\mathcal{E}/\bigcup_{j\in J}A_j'=\mathcal{F}'\vee\mathcal{G}'$. Thus, by Theorem 2.2, $\mathcal{G}'$ is binary cosimple, and by Corollary 3.7 it is determined by $\mathcal{E}/\bigcup_j A_j'$ and $\mathcal{F}'$ and hence by $\mathcal{E}$ and $\mathcal{F}$. ∎

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